Resolution Machinery*

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Abstract

This paper shows that syntax is required to represent problems, and that a computationally universal, also known as Turing complete, device is required to resolve problems. This means that problem resolution needs syntax, and needs all of it: problem resolution needs the whole syntax. So, once our ancestors achieved Turing completeness, they acquired full syntax, and their brains became resolution machines.

Introduction

Problem, and other concepts in its semantic field, such as question, doubt, uncertainty, obstacle, difficulty, resolution, decision, plan, strategy, tactic, solution, satisfaction, target, aim, goal, purpose, answer, etc., are nearly everywhere. For example, when we talk about how something should be, we are describing a wished future state, but, at least implicitly, we are also judging the current state defective, and we are suggesting to take a path that reaches the target. Every time we look for some way to go from some state to some other better state we are in a problem resolving mood.

But, in spite of 'problem' pervasiveness, or, perhaps, because of it, and to the best of my knowledge, there is not any problem theory exact enough to be used in mathematics. So we will define one from first principles. The result is presented in section "Problem Theory". This theory uses just eight concepts: problem, freedom, condition, resolution, routine, trial, analogy, and solution. The theory will be evaluated later, so for now the reader should focus in getting an accurate view of each of the eight concepts; but do not worry, we are natural problem resolvers, so they are easy to grasp.

The next section, "Problem Resolving", is the core of the paper, but for anyone familiar with computing theory the bulk of the work was already done while presenting the problem theory. The aim of this section is to translate the eight concepts of the theory to mathematics. The first conclusion is that open expressions, that is functions, are needed to translate problems to mathematics. And the last conclusion is that the whole power of Church's lambda calculus, that is a Turing complete or a computationally universal device, is needed to resolve problems. This last conclusion, that should not be surprising for computing theorists, should instead serve to demonstrate the value of the problem theory presented in the previous section.

In section "Syntax" we show, standing in the shoulders of Chomsky, that the whole power of a universal Turing machine, that is a Turing complete or a computationally universal device, is needed to generate any possible syntax. This is another unsurprising conclusion, but, together with the previous one, we get something interesting: a syntax engine is a resolution machine, because both are universal computers.

So in the last section, "Implications", we explore some consequences beyond mathematics of the equation of syntax with problem resolution that we have found via Turing completeness. And, summarizing, the two most important achievements of this paper are this equation and the problem theory.

Problem Theory

Before investigating what is needed to represent problems, we should investigate what problems are; see [Casares 1993].

* This is a draft. Any comments on it to `papa@ramoncasares.com` are welcome.

Problem

Every *problem* is made up of freedom and of a condition. There have to be possibilities and *freedom* to choose among them, because if there is only necessity and fatality, then there is neither a problem nor is there a decision to make. The different possible options could work, or not, as solutions to the problem, so that in every problem a certain *condition* that will determine if an option is valid or not as a solution to the problem must exist.

$$\text{Problem} \begin{cases} \text{Freedom} \\ \text{Condition} \end{cases}$$

Solution

A fundamental distinction that we must make is between the solution and the resolution of a problem. Resolving is to searching as solving is to finding, and please note that one can search for something that does not exist.

$$\begin{array}{c} \text{Resolving} \cdot \text{Searching} \\ \text{Solving} \cdot \text{Finding} \end{array}$$

Thus, *resolution* is the process that attempts to reach the solution to the problem, while the *solution* of the problem is a use of freedom that satisfies the condition.

$$\text{Problem} \longrightarrow \text{Resolution} \longrightarrow \text{Solution}$$

In the state-transition jargon: a problem is a state of uncertainty, a solution is a state of satisfaction, and a resolution is a transition from doubt to certainty.

We can explain this with another analogy. The problem is defined by the tension that exists between two opposites: freedom, free from any limits, and the condition, which is pure limit. This tension is the cause of the resolution process. But once the condition is fulfilled and freedom is exhausted, the solution annihilates the problem. The resolution is, then, a process of annihilation that eliminates freedom as well as the condition of the problem, in order to produce the solution.

$$\underbrace{\left.\begin{array}{c} \text{Freedom} \\ \text{Condition} \end{array}\right\}}_{\text{Problem}} \xrightarrow{\quad\text{Resolution}\quad} \text{Solution}$$

A mathematical example may also be useful in order to distinguish resolution from solution. In a problem of arithmetical calculation, the solution is a number and the resolution is an algorithm such as the algorithm for division, for example.

Resolution

There are three ways to resolve a problem: routine, trial, and analogy.

$$\text{Resolution} \begin{cases} \text{Routine} \\ \text{Trial} \\ \text{Analogy} \end{cases}$$

To resolve a problem by *routine*, that is, without reasoning, it is necessary to know the solution, and it is necessary to know that it solves that problem.

If the solution to a problem is not known, but it is known a set of possible solutions, then we can use a trial and error procedure, that is, we can try the possible solutions. To *try* is to test each possible solution until we find one that satisfies the condition. There are two tasks when we try: to test if a particular possibility

satisfies the condition, and to govern the process determining the order of the tests. There are several ways to govern the process, that is, there is some freedom in governing the trial, and so, if we also put a condition on it, for example a temporal milestone, then governing is a problem. And there are three ways to resolve a problem (*da capo*).

By *analogy* we mean to transform a problem into a different one, called question, which is usually composed of several subproblems. This works well if the subproblems are easier to resolve than the original problem. There are usually several ways to transform any problem (there is freedom), but only those transformations which result in questions that can be resolved are valid (which is a condition), so applying an analogy to a problem is a problem. There are three ways to resolve the analogy, the question, and each of its subproblems: routine, trial, and analogy (*da capo*). If we could translate a problem into an analogue question, and we could find a solution to that question, called answer, and we could perform the inverse translation on it, then we would have found a solution to the original problem.

$$\begin{array}{ccc} \text{Problem} & & \text{Solution} \\ \downarrow & & \uparrow \\ \text{Question} & \longrightarrow & \text{Answer} \end{array}$$

Problem Resolving

Once we know what the raw ingredients of problems, resolutions, and solutions are, we can now look for the means to represent and to execute problem resolutions; see [Casares 2010].

Unknown

To express a problem, we have to refer to its freedom and to its condition. To name the freedom we have to use a word that does not refer to anything, that is, it has to be a word free of meaning. For example, if the problem is that we do not know what to do, then its more direct expression in English is 'what to do?'. In this sentence, the word 'what' does not refer to anything specifically, but it is a word free of meaning and purely syntactical.

In mathematics, the typical way to express the freedom of a problem is to use the unknown x, that works the same way as the interrogative pronoun 'what'. For example, if we want to know which number is the same when it is doubled as when it is squared, we would write:

$$x? \quad 2x = x^2.$$

The condition is, in this case, the equality $2x = x^2$. Equality is a valid condition because it can be satisfied, or not.

The x is just a way to refer to something unknown, so we could use any other expedient just by indicating it with the question mark (?). This means that

$$y? \quad 2y = y^2$$

is exactly the same problem. We call this equivalence α-conversion.

It is important to note that the unknown has to be part of the condition, in order to determine if a value is a solution to the problem, or not. In the condition, the unknown x is a free variable, and therefore the condition is an open expression, that is, a function.

In summary, in order to refer to the freedom of a problem we have to use free variables, which are words free of meaning that do not refer to anything. These free words are useless by themselves, we can even substitute one for another using an α-conversion, so they have to be combined with other words to compose the condition of the problem. We will call this structure of words a sentence. All things related to the sentence, as, for example, the rules for word combination, are what is usually called syntax.

Definition

We have just seen why we need syntax to represent a problem, and why separate words are not enough. We have also seen two types of word: semantic words with some meaning, and syntactical words without any meaning. Well, although it seems impossible, there is a third type of word: defined words.

A defined word is just an abbreviation, so we could go on without them, but they are handy. That way we substitute a word for a whole expression. We can, for example, define a word to refer to a problem:

$$q := \langle x?\ \ 2x = x^2 \rangle.$$

Ordered Pair

The resolution of the problem

$$x? \quad 2x = x^2$$

can go on this way:

$$\begin{aligned} 2x &= x^2 \\ 2x - 2x &= x^2 - 2x \\ 0 &= xx - 2x \\ 0 &= (x-2)x \\ [x-2=0] &\vee [x=0] \\ [x-2+2=2] &\vee [x=0] \\ [x=2] &\vee [x=0] \\ 2 &\vee 0\,. \end{aligned}$$

So it has two solutions, two and zero. In this case the resolution was achieved by analogy transforming the problem until we found two subproblems with a known solution, $x?\ x = 2$ and $x?\ x = 0$, that we then could solve by routine.

To represent each transformation the simplest device is the ordered pair, (s, t), where s and t represents two expressions: s before being transformed, and t once it has been transformed. Note that we can use a single ordered pair to express each transformation, or the whole sequence. For example, the complete sequence can be summarized in just one ordered pair, that then describes how to resolve the problem by routine, because s is the problem and t its solution:

$$\big(x?\ \ 2x = x^2,\ 2 \vee 0\,\big)\,.$$

Function

We can also resolve by trial and error. In the trial we have to test if a value satisfies the condition, or not, and the condition is an open expression. To note mathematically open expressions, also known as functions, we will use lambda expressions, $\lambda_x\xi$, where x is the free variable, and ξ the open expression. In the case of our problem:

$$\lambda_x[2x = x^2].$$

Now, to test a particular value a, we have to bind that value a to the free variable inside the condition. We also say that we apply value a to the function $\lambda_x\xi$. In any case we write the binding this way: $\lambda_x\xi(a)$. We can abbreviate the expression naming the function, for example $f := \lambda_x\xi$, to get the typical $f(a)$. In our case, 'to test if number 2 is equal when it is doubled to when it is squared' is written $\lambda_x[2x = x^2](2)$. And to calculate if a value satisfies the condition, we replace the free variable with the binding value; this process is called β-reduction. In our case we replace x with 2 ($x \leftarrow 2$), this way:

$$\lambda_x[2x = x^2](2) \to 2.2 = 2^2 \to 4 = 4 \to \text{YES}.$$

A condition can take only two values, that we will call YES and NO. In case we want to make a difference depending on the condition, and what else we could want, then we have to follow one path when the condition is satisfied (YES), and a distinct path when the condition is not satisfied (NO). To achieve this, a command **if** is used:

$$\textbf{if}\,\langle \textit{condition}\rangle\,\textbf{then}\,\langle \textit{case}\ \text{YES}\,\rangle\,\textbf{else}\,\langle \textit{case}\ \text{NO}\rangle\,.$$

We can write down completely any trial just by using this two devices: binding values to free variables in open expressions, and a conditional command, as **if**. Suppose, for example, that we guess that the solution to our problem is one of the first four numbers, in other words, that the solution is in set $\{1, 2, 3, 4\}$, and that we want to try them in increasing order. Then we should nest four **if** commands:

$$\begin{array}{l}\textbf{if}\ \lambda_x[2x = x^2](1)\ \textbf{then}\ 1\\ \textbf{else if}\ \lambda_x[2x = x^2](2)\ \textbf{then}\ 2\\ \qquad\textbf{else if}\ \lambda_x[2x = x^2](3)\ \textbf{then}\ 3\\ \qquad\qquad\textbf{else if}\ \lambda_x[2x = x^2](4)\ \textbf{then}\ 4\\ \qquad\qquad\qquad\textbf{else}\ \text{NO}\ .\end{array}$$

Condition

A known condition is a function with two possible outcomes, the loved YES and the hated NO. This means that each time we apply it we get a YES or a NO. For example:

$$\begin{aligned}\lambda_x[2x = x^2](0) &\to \text{YES}\\ \lambda_x[2x = x^2](1) &\to \text{NO}\\ \lambda_x[2x = x^2](2) &\to \text{YES}\\ \lambda_x[2x = x^2](3) &\to \text{NO}\\ \lambda_x[2x = x^2](4) &\to \text{NO}\\ &\vdots\end{aligned}$$

What resolves definitively a problem is the inverse function of the condition of the problem. Because the inverse function just reverses the condition, from a $0 \to$ YES to a YES $\to 0$, so when we apply YES to the inverse condition we get the solutions, and when we apply NO we get the set of the no-solutions. Thus, naming f the function $\lambda_x[2x = x^2]$:

$$\begin{aligned}f^{-1}(\text{YES}) &\to \{0, 2\}\\ f^{-1}(\text{NO}) &\to \{\ldots, -2, -1, 1, 3, 4, \ldots\}.\end{aligned}$$

We can use the condition in both directions: the natural direction, when we apply a value to test if it satisfies the condition, and the opposite direction, when we want to know what values satisfy the condition. To express a problem is enough to write the condition and to indicate which are its free variables, and to resolve a problem is enough to apply the condition in the opposite direction.

It is too easy to say that the resolution of a problem is the inverse function of its condition, I have just done it. Unfortunately, it is nearly always impossible to calculate that inverse function, and in some cases the condition is unknown. So, finding a resolution to a given problem is a problem.

Finding the resolution to a problem is a problem because it has its two mandatory ingredients. There is freedom, because there are several ways to resolve a problem, and there is a condition, because not every resolution is valid, but only those that could provide the solutions to the problem. And then, being a problem, we need a resolution to find a resolution to the problem. And, of course, to find a resolution to find a resolution to the problem we need, can you guess it?, another resolution. What do you think? Better than 'convoluted' say 'recursive'.

Tree

If we know the solution of a problem, then we can apply a routine, and write it as an ordered pair, as we have already done, $\big(x?\ 2x = x^2,\ 2 \vee 0\big)$. But we can also write it as a function, using the command **if**:

$$\lambda_\pi \left[\textbf{if } \pi \equiv \langle x?\ 2x = x^2 \rangle \textbf{ then } 2 \vee 0 \textbf{ else } \text{NO}\right].$$

Here π is a free variable that we can bind to any expression, that then is compared ($\equiv$), not against the problem, but against the expression of the problem that we know how to resolve ($x?\ 2x = x^2$), and, **if** the comparison is successful, **then** we get its two solutions ($2 \vee 0$). When comparing open expressions, remember α-conversion and β-conversion.

I will not add anything new about trials, because we already know that we can express any trial just nesting command conditionals of bound functions. But remember that frequently determining the order of the tests is a problem.

By analogy we transform a problem in another problem, or problems. Most times the outcome will be more that one problem, because 'divide and conquer' is usually a good strategy for complex problems. So, in general, the resolution to a problem will be a tree, being the original problem its trunk. If we use analogy to resolve it and we get, for example, four easier subproblems, then the tree has four main branches. But, again, from each branch we can use analogy, an we get some sub-branches, or we can use routine or trial. We resolve by routine when we know the solution, so the subproblem is solved; these are the leaves of the resolution tree. Trial ...

You got lost? Don't worry, even myself get lost in this recursive tree, and to what follows the only important thing to keep in mind is one easy and true conclusion: expressions that represent resolutions to problems have a tree structure, because they describe the resolution tree. You also has to know that it is enough to allow that the elements of a ordered pair be ordered pairs to be able to build binary trees with ordered pairs.

Recursion

Resolution is a process that, when it is successful, transforms a problem into a solution. But this neat and tidy view hides a more complex resolution tree, one that is full of subproblems. Inside the tree we see partial resolutions, that are atomic transformations transforming a subproblem into a subproblem.

The solution to a problem can also be a problem or a resolution. For example, when a teacher is looking for a question to include in a test, her solution is a problem. And when an engineer is designing an algorithm to implement some general type of electrical circuits, her solution is a resolution. Note that, in this last case, a previous step could be a sub-optimal algorithm to be enhanced, and then one of the transformations would take a resolution to return a resolution; this also happens with solutions.

Any open condition is a problem, if we are interested in the values that satisfy the condition. An open condition is an open expression, and closed expressions are just open expressions with no free variables, that is, with zero free variables. In addition, you can close any open expression by binding all its free variables. So, with this in mind, we can treat open expressions as expressions. Not every open expression is an open condition, but only generalizing to full functions we can cope with the resolution to resolution transformation that the engineer above needed.

The conclusion is then that a partial resolution can start with any expression and can end with any expression. So a resolution has to be able to transform any expression into any expression.

$$\left.\begin{matrix}\text{Problem}\\ \text{Resolution}\\ \text{Solution}\end{matrix}\right\} \xrightarrow{\quad\text{Resolution}\quad} \left\{\begin{matrix}\text{Problem}\\ \text{Resolution}\\ \text{Solution}\end{matrix}\right.$$

Please note that the resolution can be the transformation, but also what is to be transformed, and what has been transformed into. We call recursivity this feature of transformations that can act on themselves.

Quoting

Recursivity needs a quoting device to indicate if an expression is referring to a transformation, or it is referring to something to be transformed.

Joke:
— Tell me your name.
— Your name.
Who is answering has understood "tell me 'your name'", which could be the case, although unlikely. In any case, quotation prevents the confusion.

Quoting is very powerful. Suppose that I write:

En cuanto llegué, el inglés me dijo: "I was waiting you to tell you something. . . .

⋮

[a three hundred pages story in English]

⋮

. . . And even if you don't believe me, it is true". Y se fue, sin dejarme decirle ni una sola palabra.

Technically, I would have written a book in Spanish.

Technically, using the quoting mechanism, all symbolic languages are one. As we learned from Chomsky, there is only one language; see [Chomsky 2000]. And, as in the Spanish book, even if you don't believe me, it is true.

Requirements

We are ready to summarize the capacities necessary to represent and to execute problem resolutions. We will call the resulting system *syntax*, although the justification for using this name will be provided after this section.

Syntax uses three types of words.

- Semantic words: These are the link to semantics.
- Syntactic words: These are the core of syntax, so in this section we will note its name, between parenthesis, in Lisp dialect Scheme; see [Abelson and Sussman 1985].
- Defined words: These are just handy abbreviations.

Syntax needs, then, a dictionary to keep the abbreviations, and operations to add a dictionary entry defining an abbreviation (`define`), and also to modify or to delete an entry (`set!`).

Syntax atoms are the words, that can be used to compose sentences. A sentence has a tree structure, so inside a sentence we can find other sentences. Both words and sentences are called expressions. Because there are two kind of expressions, words and sentences, syntax needs a condition to test the type of an expression (`atom?`). Syntax needs operations both to compose sentences from expressions (`cons`), and to take the expressions from the sentences where they are in (`car`, `cdr`).

Syntax also needs a condition to check if two expressions are equal (`equal?`). Most times, this depends eventually on semantic equality, but in any case syntax has to work its part, for example honoring α and β-conversions on open expressions.

Sentences can be open expressions (`lambda`), also known as functions. These functions are recursive, meaning that you can apply any expression to the function using β-reduction, or, in other words, that you can bind any expression to any of the free variables of the open expression. Because of this feature, functions can be applied on themselves, and a quoting device is needed to indicate that the expression is not referring to a function (`quote`).

Finally, syntax need a conditional command so its behaviour can depend on the result of a condition (`cond`).

Resolution Machine

The requirements in the previous section define a limited Lisp, that, nevertheless, is a superset of Church's lambda calculus; as defined in [Church 1935]. As Turing proved that lambda calculus is computationally universal, see [Turing 1936], the conclusion is that to resolve problems a Turing complete device is required. Or saying the same thing in other words, a resolution machine is a universal computer.

$$\text{Resolution Machine} = \text{Universal Computer}$$

Syntax

Grammar

Chomsky presented, in [Chomsky 1959], a hierarchy of grammars. A *grammar* of a language is a device which is capable of enumerating all the language sentences. And, in this context, *language* is the (usually infinite) set of all the valid sentences.

At the end of SECTION 2 in that paper, we read: "A type 0 grammar (language) is one that is unrestricted. Type 0 grammars are essentially Turing machines". At the beginning of SECTION 3, we find two theorems.

THEOREM 1. For both grammars and languages, type 0 $\supseteq$ type 1 $\supseteq$ type 2 $\supseteq$ type 3.

THEOREM 2. Every recursively enumerable set of strings is a type 0 language (and conversely).

Then THEOREM 2 is explained: "That is, a grammar of type 0 is a device with the generative power of a Turing machine."

From the two theorems we can deduce three corollaries:

COROLLARY 1. The set of all type 0 grammars (languages) is equal to the set of all grammars (languages).
This is because, according to THEOREM 1, type 0 is the superset of all grammars (languages).

COROLLARY 2. For each Turing machine there is a type 0 grammar (and conversely).
This is equivalent to THEOREM 2, but in terms of grammars (devices) instead of languages (sets).

COROLLARY 3. For each Turing machine there is a grammar (and conversely).
This results by applying COROLLARY 1 to COROLLARY 2.

Syntax Engine

We will call any device that can generate any possible language a *syntax engine*. For each Turing machine there is a grammar and for each grammar there is a Turing machine, see COROLLARY 3, and therefore a syntax engine has to be able to behave as any Turing machine, that is, it has to be Turing complete. In other words, to generate any possible language a computationally universal device is required. Or more concisely: a syntax engine is a universal computer.

$$\text{Syntax Engine} = \text{Universal Computer}$$

We saw that a resolution machine is a universal computer, and that a universal computer is a syntax engine, so a resolution machine is a syntax engine.

$$\text{Resolution Machine} = \text{Universal Computer} = \text{Syntax Engine}$$

Syntax Definition

We have chosen 'syntax engine' rather than 'language engine' to name a device that can generate any possible 'language' because 'language' in [Chomsky 1959] refers only to 'syntax'. It does not refer to semantics, because meanings are not considered, nor to pragmatics, because intentions are not considered.

So now we will provide the definition of syntax that follows from [Chomsky 1959] after adjusting for this deviation towards language: *syntax* consists of transformations of strings of symbols, irrespective of the symbols meanings, but according to a finite set of well defined rules; so well defined as a Turing Machine is. This definition of syntax is very general and includes natural languages syntax, just replacing symbols with words and strings with sentences, but it also includes natural languages morphology or phonology, taking then subunits of words or sounds.

As an aside, please note that this definition of syntax manifests, perhaps better than other definitions, what we will call the syntax purpose paradox: syntax, being literally meaningless, should be purposeless. It is easy to underestimate the value of some mechanical transformations of strings of symbols.

Implications

Syntax and Problems

We could equate problem resolution with syntax, because problem resolution is not possible without syntax, but mainly because the requirement that problem resolution impose is universal computability, which is the maximum requirement. So problem resolution needs syntax, and needs all of it: problem resolution needs the whole syntax.

By equating syntax to problem resolution we have solved the syntax purpose paradox: when a brain become a syntax engine, then that brain become a resolution machine. So syntax is meaningless but very useful and its purpose is to resolve problems.

Syntax and problem resolution converge at the end, but not at the beginning. This means that a full recursive syntax engine is equal to a whole resolution machine, but also that a minimal syntax has little, if any, relation to problem resolution. For example, the first step of syntax could be very simple: just put the agent before any other word. This simple expedient by itself would prevent efficiently some ambiguities, explaining who did and who was done something.

Syntax recursivity is credited with making possible the infinite use of finite means, which is obviously true. But our wanderings in the problem resolution realm have shown us that there are other capacities provided by recursive syntax that are, at least, as important as infinity; infinity that, in any case, we cannot reach. My own two favorites are functions, that is, open expressions with free variables, and conditionals, as 'if'.

Syntax and Evolution

These results should help us to reevaluate syntax. In language evolution there are two main positions regarding syntax; see [Kenneally 2007]. The continuist side defends a more gradual evolution, where syntax is just a little step forward that prevents some ambiguities and makes a more fluid communication; see [Bickerton 2009]. For the other side, led by Chomsky, syntax is a hiatus that separates our own species from the rest; see [Hauser, Chomsky, and Fitch 2002]. What position should we take?

The convergence until fusion of syntax and problem resolution explains that, once our ancestors achieved Turing completeness, they acquired full syntax, and their brains became resolution machines. Then they could see a different the world. How much different? Very much.

Seeing the world as a problem to resolve implies that we look for the causes that are hidden behind the apparent phenomena. But it is more than that. Being able to calculate resolutions inside the brain is thinking about targets, aims, purposes, intentions. This is more than foreseeing the future, it is building the future to our own will. Talking about building, what about tools? A tool is the physical realization of a resolution. So, with syntax we can design tools, and even design tools to design tools.

Thinking the world as a problem to resolve implies also that we calculate how to use freedom in order to achieve our objectives. So, yes, we are free because of syntax; see more on this in [Casares 2012].

Summarizing, the identification of syntax with problem resolution explains why syntax, being so little thing, has made us so different. And, although syntax is more than just one operation, so we have probably acquired syntax in more than one evolutionary step, my vote goes, anyway, to ... Chomsky!

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